Instantons and emergent AdS$_3\times$S^3 geometry

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ABSTRACT: Two dimensional $\mathcal{N} = (4, 4)$ gauge theories flow to interacting superconformal field theories on their Higgs branch. We examine worldsheet instantons in these theories through the eyes of a D-brane construction. The effective instanton partition function is shown to reveal an emergent background $AdS_3 \times S^3$ geometry.

Dedicated to the memory of John Brodie and Andrew Chamblin
and the roadtrip to Roswell.

Contents

1. Introduction

The notion of gravity as an emergent force holds a certain appeal. The no-go theorem of Weinberg and Witten [1] presents a hurdle that any attempt to dynamically generate a generally covariant theory of gravity must negotiate, and several examples which evade the assumptions of the theorem are now known. The most compelling of these is the celebrated AdS/CFT correspondence [2] in which both gravity and the bulk of spacetime appear dynamically.

The emergence of a bulk geometry from field theoretic origins is a strong coupling phenomenon that is not easy to see. Indeed, it is hard to determine from first principles which field theories have a string dual. A window is provided by BPS solitons. Within the framework of the AdS_5/CFT_4 correspondence, it is Yang-Mills instantons which provide a natural probe of the emergent geometry where they are represented by D-instantons which may roam the higher dimensional space [3, 4]. It was shown by Dorey et. al. [5] that the partition function of instantons in $SU(N)$ $\mathcal{N} = 4$ super Yang-Mills coincides with that of the D-instanton. Most notably, the effective large N moduli space of Yang-Mills instantons in this theory is $AdS_5 \times S^5$. The radial direction of AdS_5 arises from the scaling mode of the instanton while the internal S^5 space arises from a correct treatment of the fermionic zero modes in the large N limit.

The techniques developed in [5] were subsequently extended to study the emergent geometry in other four-dimensional gauge theories. These include $\mathcal{N} = 2$ superconformal theories [6], orbifold theories [7], $SO(N)$ and $Sp(N)$ gauge groups [8 – 10], field theories on their Coulomb branch [11] and the β-deformation of the $\mathcal{N} = 4$ theory [12].

The purpose of this paper is to employ similar techniques to the case of two-dimensional superconformal field theories (SCFTs). In general, a geometry with AdS_3 factor of radius L is dual to a 2d conformal field theory with central charge $c = 3L/2G_N$, with G_N the 3d

Newton constant [13]. The most well studied examples are the $\mathcal{N} = (4,4)$ SCFTs arising on the D1-D5 system wrapped on a 4-manifold $X \cong K3$ or $\mathbf{T}^4$ for which the dual geometry is $AdS_3 \times S^3 \times X$ (see [14] for more details). The natural probes of the emergent geometry are sigma-model instantons of the SCFT. Indeed, it was shown in [15] that these correspond to D-instantons roaming the six-dimensional supergravity background $AdS_3 \times S^3$.

Here we focus on simpler conformal field theories arising on the Higgs branch of $\mathcal{N} = (4,4)$ gauge theories which do not have a known AdS dual, in particular 2d SQCD with $N_f > N_c$. As we shall review, sigma-model instantons only arise when the singularity of the Higgs branch is lifted through the introduction of a Fayet-Iliopoulos term in the gauge theory. There are some subtleties with sigma-model instantons which do not appear in the case of four-dimensional Yang-Mills instantons. Among them is the fact that the scaling mode suffers a logarithmic divergence for conformal field theories living on the plane. We evade this problem by working with the D-brane realization of these instantons presented in [16], for which this divergence is cured. Our primary result is that, in the limit of large central charge, the instantons naturally reside in an emergent $AdS_3 \times S^3$ geometry.

The organization of the paper is as follows: sections 2 and 3 contain review material on $\mathcal{N} = (4,4)$ SCFTs on the Higgs branch and their instantons respectively. The meat of the paper is in section 4. Here we examine the partition function of the D-brane realization of sigma-model instantons and show how the effective instanton moduli space becomes $AdS_3 \times S^3$.

2. The conformal field theory of the Higgs branch

Aspects of $\mathcal{N} = (4,4)$ gauge theories in $d = 1+1$ dimensions were discussed in [17–19] and we restrict ourselves to a brief summary of the relevant points. These theories can be thought of as the dimensional reduction of $\mathcal{N} = (1,0)$ theories in six dimensions. The $\mathrm{SU}(2)_R$ R-symmetry that is enjoyed by all theories with eight supercharges is enhanced upon dimensional reduction to

$$R = \mathrm{SU}(2)_R \times \mathrm{SU}(2)_l \times \mathrm{SU}(2)_r \tag{2.1}$$

We work with gauge group G. The vector multiplet consists of a gauge potential A, four real scalars V transforming under R as $(\mathbf{1}, \mathbf{2}, \mathbf{2})$, together with four Weyl fermions. There is also a triplet of auxiliary fields D transforming as $(\mathbf{3}, \mathbf{1}, \mathbf{1})$. The hypermultiplet also contains four real scalars H lying in the $(\mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1})$ representation and an additional four Weyl fermions. The scalar potential of the theory is schematically

$$\mathbb{V}_{2d} = \frac{1}{2e^2} \mathrm{Tr} \left([V, V]^2 - D^2 \right) + \sum_{\mathrm{hypers}} (VH)^2 + DHH \tag{2.2}$$

where e^2 is the gauge coupling which, in $d = 1+1$, has scaling dimension $[e^2] = 2$. Solutions to the classical vacuum equations $\mathbb{V}_{2d} = 0$ fall roughly into two classes. The Coulomb branch has $H = 0$ and is parameterized by commuting matrices V. In contrast, the Higgs branch has $V = 0$ and is parameterized by H subject to gauge identification

and the condition $D = 0$. There can also be mixed branches. Quantum mechanically the ground state wavefunction spreads over these classical moduli spaces of vacua.

In the far infra-red, a limit which requires $e^2 \to \infty$, our $d = 1+1$ gauge theory is expected to flow to an interacting $\mathcal{N} = (4,4)$ SCFT. In fact, it is expected to flow to two decoupled superconformal field theories: one arises on the Higgs branch and the other on the Coulomb branch. Classically these two branches are connected at $V = H = 0$, but quantum effects can be shown to push this point to infinity in field space through the generation of "throat"-like wavefunction renormalization [17, 20, 19]. The simplest argument to show that the two branches indeed decouple uses the R-symmetry in the $\mathcal{N} = (4,4)$ superconformal algebra [18] which includes both left-moving and right-moving SU(2) R-symmetries. Since a symmetry which rotates scalar fields cannot be split into left- and right-moving pieces, the $\mathrm{SU}(2)_R$ symmetry of (2.1) cannot be part of the R-symmetry of the Higgs branch SCFT, while the $\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r$ R-symmetry cannot be part of the R-symmetry of the Coulomb branch SCFT.

In this paper we will focus on the conformal field theory of the Higgs branch, for which the $\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r$ R-symmetry of the gauge theory happily descends to the SCFT. The central charge of the theory coincides with the dimension of the Higgs branch: $\hat{c} = 2(n_H - n_V)$ where n_H and n_V are the number of hypermultiplets and vector multiplets respectively. This central charge can be confirmed by computing the anomaly in the two point function of R-symmetry currents [18].

The Higgs branch of (2.2) has a singularity at $H = 0$ where gauge transformations do not act freely. If the gauge group G includes abelian factors, there exists a set of marginal deformations which resolve this singularity. From the perspective of the gauge theory these arise through the addition of a triplet of Fayet-Iliopoulos (FI) parameters r transforming under the R-symmetry as $(\mathbf{3}, \mathbf{1}, \mathbf{1})$, together with a theta term θ,

$$\mathrm{Tr}_G \left(rD + i\theta F_{12} \right) \tag{2.3}$$

As we will review in section 3, the geometric resolution of the singularities also introduces worldsheet instantons into the Higgs branch theory in the guise of gauge theoretic vortices.[1] Before turning to the instantons, we first describe the specific $\mathcal{N} = (4,4)$ gauge theory of interest: 2d SQCD.

2.1 2d SQCD

Our primary focus will be on 2d SQCD with gauge group $G = \mathrm{U}(N_c)$ and N_f hypermultiplets transforming in the fundamental representation of G. The theory exhibits an $\mathrm{SU}(N_f)$ flavor symmetry. We write the hypermultiplets H as a doublet of complex scalars, $H = (Q, \tilde{Q}^\dagger)$, so that Q transforms as $(\mathbf{N}_c, \bar{\mathbf{N}}_f)$ and $\tilde{Q}$ as $(\bar{\mathbf{N}}_c, \mathbf{N}_f)$ under $G \times \mathrm{SU}(N_f)$.

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[1]The metric on the Higgs branch does not receive quantum corrections, suggesting that the singular point $H = V = 0$ may lie at finite distance in field space. Nonetheless, there is another description of the physics near the singularity which exhibits a "throat"-like structure [20, 19] with a continuous spectrum of dimensions for primary operators. In this description, the marginal deformation which resolves the singularity gives rise to a Liouville potential. It would be interesting to study the role of instantons in the throat.

After a suitable $SU(2)_R$ rotation of the FI parameters, the triplet of D-term equations $D = 0$ decomposes into the F-flatness condition

$$\sum_{a=1}^{N_f} Q_a \tilde{Q}_a = 0 \qquad (2.4)$$

together with the D-flatness condition

$$\sum_{a=1}^{N_f} Q_a Q_a^\dagger - \tilde{Q}_a^\dagger \tilde{Q}_a = r \qquad (2.5)$$

where we have explicitly displayed the flavor index $a = 1, \ldots, N_f$, while both (2.4) and (2.5) are adjoint valued in G. The Higgs branch $\mathcal{M}$ is defined as the solutions to these equations modulo the gauge action $Q \to UQ$, $\tilde{Q} \to \tilde{Q}U$, with $U \in G$. Solutions to (2.5) only exist for $N_f \geq N_c$ and the complex dimension of the Higgs branch is given by

$$\hat{c} = 2N_c(N_f - N_c) \qquad (2.6)$$

The Higgs branch is the cotangent bundle of the Grassmannian: $\mathcal{M} \cong T^\star G(N_c, N_f)$. The zero section $G(N_c, N_f)$, with radius $\sqrt{r}$, lies at $\tilde{Q} = 0$.

Since the Higgs branch is known to flow to an interacting conformal field theory, one may wonder whether there is a gravity dual in the limit of large central charge $\hat{c}$. None is known. However, 2d SQCD may be realized on the worldvolume of D-branes via the usual Hanany-Witten set-up, suspending N_c D2-branes between two NS5-branes in type IIA string theory with the role of the flavors is played by N_f D4-branes [21, 22]. Our main result in section 4 is that a suitable probe of the IIA brane configuration does indeed reveal an $AdS_3 \times S^3$ geometry that one may expect for the gravity dual of an $\mathcal{N} = (4, 4)$ SCFT.

3. Worldsheet instantons on the Higgs branch

Turning on the FI parameter r resolves the singularity on the Higgs branch by blowing up a two-cycle. This introduces worldsheet instantons into the conformal field theory on the Higgs branch which wrap this two-cycle. From the perspective of the gauge theory these instantons arise as Nielsen-Olesen vortices [23] (sometimes referred to in the soliton literature as "semi-local" vortices [24, 25]). In what follows, we shall use the word "instanton" and "vortex" interchangeably.

The instanton number is labelled by the first Chern class, the integral of the magnetic flux F_{12} over the Euclidean plane,

$$-\frac{1}{2\pi} \int \mathrm{Tr}\, F_{12} = k \in \mathbf{Z} \qquad (3.1)$$

In a semiclassical calculation, one must integrate over the Higgs branch weighted by the ground state wavefunction. However, from the perspective of vortices, not all classical vacua on the Higgs branch are created equal: BPS vortices, which descend to holomorphic sigma model instantons in the infra-red limit, exist only on the submanifold of the Higgs branch

which emerges from the singularity at finite FI parameter r. In the specific case of SQCD described in the previous section, this means that BPS vortices only exist on the $G(N_c, N_f)$ zero section of the Higgs branch, defined by $\tilde{Q} = 0$. These vacua are distinguished by the spontaneous symmetry breaking of the gauge and flavor groups:

$$\mathrm{U}(N_c) \times \mathrm{SU}(N_f) \to [\mathrm{U}(N_c)_{\mathrm{diag}} \times \mathrm{U}(N_f - N_c)]/\mathrm{U}(1) \tag{3.2}$$

For $k > 0$, the first order vortex equations require $\tilde{Q} = V = 0$ and

$$F_{12} = e^2 \left(\sum_{a=1}^{N_f} Q_a Q_a^\dagger - r \right) \quad , \quad \mathcal{D}_z Q_a = 0 \tag{3.3}$$

with $z = x^1 + ix^2$. Vortices with $k < 0$ are related by a parity transformation. Bosonic solutions to (3.3) have action

$$S_{\mathrm{inst}} = 2\pi(r + i\theta)k \tag{3.4}$$

We denote the moduli space of solutions to the vortex equations as $\mathcal{V}$. The moduli space has real dimension $\dim(\mathcal{V}) = 2kN_f$. For $k = 1$ and $N_f = N_c$ — the case where there is no Higgs branch — there are two translational zero modes dictating the position of the vortex in the plane. A further $2(N_c - 1)$ orientational modes specify how the vortex sits in surviving $\mathrm{U}(N_c)_{\mathrm{diag}}$ group of (3.2) [16, 26]. All these modes have finite norm. The vortices are exponentially localized objects, with a fixed size $\rho \sim 1/e\sqrt{r}$. In the infra-red $e^2 \to \infty$ limit they become singular, point-like objects.

When $N_f > N_c$, a true Higgs branch exists and the vortices have a qualitatively different nature. The massless fields of the Higgs branch may now be excited around the background of the vortex resulting in a power-law tail specified by a size modulus $\rho \geq 1/e\sqrt{r}$. In the limit $e^2 \to \infty$, the vortices no longer become singular, but descend to the sigma model instantons on the Higgs branch. In this limit, the scaling modulus of the instanton ρ arises as a Goldstone mode from broken conformal invariance. A further $2(N_f - N_c) - 1$ orientational modes for these massless fields arise as Goldstone modes of the $\mathrm{U}(N_f - N_c)$ factor of (3.2).

The extra modes of the semi-local vortex bring with them an extra problem: they are non-normalizable for theories defined on $\mathbf{R}^2$. This well known fact about sigma-model instantons [27] is not ameliorated by the transition to finite e^2 [28, 29]. (Indeed, turning on finite e^2 changes the vortex solution only at short distance, while the divergent norm is an infra-red issue). There are a number of ways to regulate this divergence:

- One may define the field theory on the Riemann sphere with radius R. The normalization of the scaling and orientation modes now has an overall factor $\sim \log(R/\rho)$. It would certainly be interesting to reconsider the calculation of section 4 in this case.

- One may introduce distinct masses m for the fundamental hypermultiplets. Classically this lifts the Higgs branch. It also lifts the scaling and orientation modes, endowing them with a mass m and regulating their norm as $\sim \log(1/m\rho)$ [29]. The effect on the sigma-model instantons is similar to the effect of moving onto the Coulomb

branch on a Yang-Mills instanton: it causes the instanton to shrink to zero size. A computation of the emergent geometry of the Coulomb branch of $\mathcal{N} = 4$ super Yang-Mills was performed in [11].

- One may realize the sigma-model instantons as D-branes in the Hanany-Witten type set-up [16]. While the moduli space $\mathcal{V}$ of the instanton and D-brane coincide [16] (see [29] for a recent discussion specifically addressing issues for semi-local vortices), the metrics differ and, most notably, the logarithmic divergences are rendered finite.

In this paper we focus on the latter of these regulators and show how the D-brane realization of sigma-model instantons sees an emergent background geometry. This has the disadvantage that the computation is not purely field theoretic: we are working in the D-brane regime, rather than the sigma-model limit. The advantage is that the D-brane realization of the instanton partition function takes a convenient and familiar form as we now review.

3.1 A D-brane realization of instantons on the Higgs branch

A D-brane realization of vortices in eight super-charge SQCD was presented in [16] using Hanany-Witten type brane configurations. In the present context, this consists of D2-branes suspended between parallel NS5-branes, with D4-branes providing the hypermultiplets as shown in the figure [21, 22]. Separating the NS5-branes in the direction out of the page induces the FI parameter, forcing the 2d gauge theory onto the Higgs branch. The instantons are stretched Euclidean D0-branes as depicted.
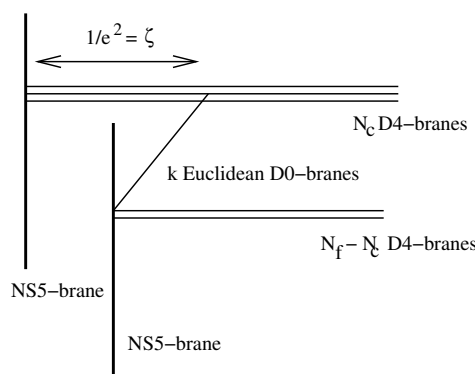

Figure 1: The D-brane construction.

The worldvolume theory on the D0-branes is a $d = 0 + 0$ matrix model with $\mathcal{N} = (2, 2)$ super-symmetry (i.e. the complete dimensional reduction of theory with $\mathcal{N} = 1$ supersymmetry in $d = 3 + 1$). We refer to this as the "vortex matrix model"; it consists of the following fields [16]

- U(k) vector multiplet, containing four real scalars X_a, $a = 1, 2, 3, 4$ and a doublet of complex Grassmannian parameters λ_α, $\alpha = 1, 2$.

- N_c chiral multiplets transforming in the fundamental **k** representation, each containing complex scalars ϕ_i, $i = 1, \ldots, N_c$ and $2N_c$ complex fermions $\mu_{i\alpha}$.

- $N_f - N_c$ chiral multiplets transforming in the anti-fundamental $\bar{\mathbf{k}}$ representation, containing complex scalars $\tilde{\phi}_r$, $r = 1, \ldots, N_f - N_c$ and $2(N_f - N_c)$ complex fermions $\tilde{\mu}_{i\alpha}$.

- A single chiral multiplet transforming in the adjoint representation of U(k). This contains a complex scalar a, together with a complex doublet of fermions χ_α.

The gauge coupling constant g^2 of the vortex matrix model is taken to infinity. The only parameter in the theory is a FI parameter ζ, inherited from the 2d gauge coupling e^2,

$$\zeta = \frac{2\pi}{e^2} \tag{3.5}$$

The full action will be presented shortly in its full indexical glory. Here we restrict attention to the U(k) valued D-flatness condition which, since we work in the limit $g^2 \to \infty$, is strictly imposed in the matrix model

$$[a, a^\dagger]^m_{\ n} + \sum_{i=1}^{N_c} \phi^m_i \phi^{\dagger i}_n - \sum_{r=1}^{N_f - N_c} \tilde{\phi}^{m\dagger}_r \tilde{\phi}^r_n = \zeta \delta^m_{\ n} \tag{3.6}$$

where $m, n = 1, \dots, k$ are gauge indices. Solutions to (3.6), modulo U(k) gauge transformations, define the vacuum manifold $\mathcal{V}$ of the vortex matrix model. We have $\dim(\mathcal{V}) = 2kN_f$ and $\mathcal{V}$ is identified with the vortex moduli space [16]. However, as promised, the metric on the Higgs branch $\mathcal{V}$, inherited through the usual Kähler quotient construction from the flat metric on $\mathbf{C}^{k(N_f+k)}$, does not suffer a divergent norm. Despite the differences in the metric, the vortex theory is expected to capture the BPS properties of field theoretic vortices, an assertion which has passed a number of checks [30, 31, 29].

For finite e^2 we have $\zeta \neq 0$ and the U(k) gauge transformations are freely acting on solutions of (3.6), ensuring that $\mathcal{V}$ is smooth as befits the vortex moduli space. However, in the $e^2 \to \infty$ infra-red limit of the 2d gauge theory, we have $\zeta = 0$ and $\mathcal{V}$ develops a singularity. This reflects the existence of singular ("small") $\rho \to 0$ instantons in the 2d sigma model. In the following we shall be interested in the infra-red SCFT on the 2d Higgs branch, and correspondingly focus on the vortex theory with $\zeta = 0$.

4. The instanton partition function

In this section we study the vortex matrix model in more detail, following closely the calculation of [5]. The cultured reader may notice the similarity between the vortex theory described above and the ADHM matrix model of Yang-Mills instantons. Indeed, the vortex moduli space can be shown to be a complex submanifold of the moduli space of SU(N_f) Yang-Mills instantons [16]. This correspondence motivated the present work and allows us to transfer the techniques developed in [5] for four-dimensional gauge theories to the present case of two-dimensional sigma-models.

The full action of the vortex matrix model can be written as the sum of three terms,

$$S_\Phi = \sum_{i=1}^{N_c} \left(\phi^{\dagger i} X_a X_a \phi_i + \bar{\mu}^i_{\dot\alpha} X_a \bar{\sigma}^{\dot\alpha\alpha}_a \mu_{i\alpha} + i\sqrt{2}\phi^{\dagger i}\lambda^\alpha \mu_{i\alpha} - i\sqrt{2}\bar{\mu}^i_{\dot\alpha}\bar{\lambda}^{\dot\alpha}\phi_i - \phi^{\dagger i} D \phi_i \right)$$

$$S_{\tilde\Phi} = \sum_{r=1}^{N_f - N_c} \left(\tilde{\phi}^r X_a X_a \tilde{\phi}^\dagger_r + \tilde{\mu}^{r\alpha} X_a \sigma^a_{\alpha\dot\alpha} \bar{\tilde{\mu}}^{\dot\alpha}_r - i\sqrt{2}\tilde{\mu}^{r\alpha}\lambda_\alpha \tilde{\phi}^\dagger_r + i\sqrt{2}\tilde{\phi}^r \bar{\lambda}_{\dot\alpha} \bar{\tilde{\mu}}^{\dot\alpha}_r + \tilde{\phi}^r D \tilde{\phi}^\dagger_r \right)$$

$$S_A = \mathrm{Tr}_k \left(-|[X_a, a^\dagger]|^2 - [X_a, \bar\chi_{\dot\alpha}]\bar{\sigma}^{\dot\alpha\alpha}_a \chi_\alpha + i\sqrt{2}\bar\chi_{\dot\alpha}[a, \bar\lambda^{\dot\alpha}] + i\sqrt{2}[a^\dagger, \lambda^\alpha]\chi_\alpha + D[a, a^\dagger] \right)$$

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Here only U(k) gauge indices have been left implicit. All others indices are explicitly displayed to expose the large global symmetry group F

$$F = \mathrm{U}(1)_R \times \mathrm{U}(1)_E \times \mathrm{Spin}(4)_R \times [\mathrm{U}(N_c) \times \mathrm{U}(N_f - N_c)]/\mathrm{U}(1) \qquad (4.1)$$

Of these, $\mathrm{U}(1)_E$ acts only on the adjoint chiral multiplet fields a and χ, and arises from Euclidean rotations of the instanton on the plane $\mathbf{R}^2$. The non-abelian $[\mathrm{U}(N_c) \times \mathrm{U}(N_f - N_c)]/\mathrm{U}(1)$ factor is identified with the corresponding factor in (3.2), with the quotient U(1) part of the U(k) gauge group. Finally, the $\mathrm{U}(1)_R \times \mathrm{Spin}(4)_R \cong \mathrm{U}(1)_R \times \mathrm{SU}(2)_l \times \mathrm{SU}(2)_r$ is identified with the 2d R-symmetry (2.1), where $\mathrm{SU}(2)_R \to \mathrm{U}(1)_R$ through the introduction of the FI parameter. The transformation of the various fields under the U(k) gauge group and the various flavor groups is shown in the table.

	U(k)	U(1)$_R$	U(1)$_E$	Spin(4)	U(N_c)	U($N_f - N_c$)
X	**adj**	0	0	$(\mathbf{2},\mathbf{2})$	$\mathbf{1}$	$\mathbf{1}$
λ	**adj**	+1	0	$(\mathbf{1},\mathbf{2})$	$\mathbf{1}$	$\mathbf{1}$
ϕ	**k**	+2	0	$\mathbf{1}$	$\bar{\mathbf{N}}_c$	$\mathbf{1}$
μ	**k**	+1	0	$(\mathbf{1},\mathbf{2})$	$\bar{\mathbf{N}}_c$	$\mathbf{1}$
$\tilde{\phi}$	$\bar{\mathbf{k}}$	+2	0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{N_f - N_c}$
$\tilde{\mu}$	$\bar{\mathbf{k}}$	+1	0	$(\mathbf{1},\mathbf{2})$	$\mathbf{1}$	$\mathbf{N_f - N_c}$
a	**adj**	0	+1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
χ	**adj**	-1	+1	$(\mathbf{1},\mathbf{2})$	$\mathbf{1}$	$\mathbf{1}$

We have included here only the unbarred fermions; the barred fermions transform in the $(\mathbf{2}, \mathbf{1})$ representation of $\mathrm{Spin}(4)_R$ and in the conjugate representation of the other symmetry groups.

We are interested in evaluating the instanton partition function

$$\mathcal{Z}_{k,N_c,N_f} = \frac{1}{\mathrm{Vol}(\mathrm{U}(k))} \int \exp - \left(S_\Phi + S_{\tilde{\Phi}} + S_A \right) , \qquad (4.2)$$

where the integration is over all fields listed in the table, together with their complex conjugates as well as the auxiliary field D. Integrating over the latter gives the D-flatness condition (3.6). Note that we have also stripped off the obvious constant factor of $\exp(-S_{\mathrm{inst}})$ given in (3.4).

The instanton contribution to a given correlator $\langle \mathcal{O} \rangle$ is evaluated by inserting $\mathcal{O}_{\mathrm{inst}}$ into the partition function $\mathcal{Z}_{k,N_c,N_f}$ where $\mathcal{O}_{\mathrm{inst}}$ is the operator $\mathcal{O}$ evaluated on the instanton background. Since $\mathcal{O}_{\mathrm{inst}}$ depends on the collective coordinates of the instanton (i.e. the various fields of the vortex matrix model) we obviously don't wish to perform all integrations in $\mathcal{Z}_{k,N_c,N_f}$ before inserting $\mathcal{O}$; rather we will perform a subset of the integrations to yield an effective partition function which can subsequently be used to evaluate correlators which are both gauge and flavor singlets.[2] As we shall see, our effective partition function will be valid in the limit of large N_c and $N_f - N_c$, and hence large central charge $\hat{c}$.

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[2]This differs from the 4d calculation of [5], where only the requirement of gauge invariance is needed. It's worth noting however that in the most studied example of AdS_3/CFT_2, the would-be flavor symmetry on the D5-brane becomes gauged once it is compactified on $X = \mathbf{T}^4$ or $K3$.

Let us start by introducing singlet fields under the $U(N_c)$ and $U(N_f - N_c)$ global symmetries of the matrix model. We define

$$W_n^m = \sum_{i=1}^{N_c} \phi_i^m \phi_n^{\dagger\, i} \quad , \qquad \tilde{W}_n^m = \sum_{r=1}^{N_f - N_c} \tilde{\phi}_r^{\dagger\, m} \tilde{\phi}_n^r \tag{4.3}$$

We are trading $2kN_c$ ϕ variables for $2k^2$ W variables (and similarly for $\tilde{\phi}$). When $k \geq N_c$, this isn't an improvement; generically orientated instantons fill the entire $SU(N_c)$ group space and we should stick with the ϕ variables. In contrast, when $k < N_c$, the instantons rattle around inside the group space, filling at most a $SU(N_c - k)$ subgroup. In this situation, the partition function picks up a volume factor from the integration over the $SU(N_c)/SU(N_c - k)$ coset space. The Jacobian factors that arise from changing from $\phi, \tilde{\phi}$ to $W, \tilde{W}$ variables were computed in [5], yielding

$$\int d\phi d\tilde{\phi} = C_{k,N_c} C_{k,N_f-N_c} \int dW d\tilde{W} \ (\det_k W)^{N_c-k} (\det_k \tilde{W})^{N_f-N_c-k} . \tag{4.4}$$

where the coefficients, which arise from the volume of the coset spaces, are given by

$$C_{k,N} = \frac{(2\pi)^{Nk - k(k-1)/2}}{\prod_{i=1}^{k}(N-i)!} \tag{4.5}$$

We may eliminate one of these integrals by imposing the D-flatness constraint (3.6) which is now linear in the $W, \tilde{W}$ variables, reading: $W - \tilde{W} = [a, a^\dagger]$.

We now turn to the fermions. The action $S = S_\Phi + S_{\tilde{\Phi}} + S_A$ contains no quadratic terms for the vector multiplet fermions λ, ensuring that they each act as a Grassmannian Lagrange multiplier, imposing constraints on the remaining fields:

$$[\chi^\alpha, a^\dagger]_n^m + \sum_{i=1}^{N_c} \mu_i^{\alpha m} \phi_n^{\dagger\, i} - \sum_{r=1}^{N_f-N_c} \tilde{\phi}_r^{\dagger\, m} \tilde{\mu}_n^{\alpha r} = 0 \tag{4.6}$$

Solutions to the complex constraint equations (4.6), modulo $U(k)$ gauge transformations, describe the physical fermionic zero modes of the instanton. We wish to mimic the procedure just performed in the bosonic sector, integrating out the superpartners of the gauge and flavor orientation modes. Schematically these are the fermions μ and $\tilde{\mu}$ satisfying $\phi\bar{\mu} = \tilde{\phi}^\dagger \tilde{\mu} = 0$. More precisely, we may decompose the fermionic variables as

$$\mu_i^{\alpha m} = \theta_n^{\alpha m} \phi_i^n + \nu_i^{\alpha m} \quad , \qquad \tilde{\mu}_m^{\alpha r} = \tilde{\phi}_n^r \tilde{\theta}_m^{\alpha n} + \tilde{\nu}_m^{\alpha r} \tag{4.7}$$

where ν and $\tilde{\nu}$ are the superpartners of the orientation modes, satisfying

$$\sum_{i=1}^{N_c} \phi_i^m \bar{\nu}_n^{\dot{\alpha} i} = 0 \quad \text{and} \quad \sum_{r=1}^{N_f} \tilde{\phi}_r^{\dagger\, m} \tilde{\nu}_n^{\alpha r} = 0 \tag{4.8}$$

In terms of these new variables, the constraints (4.6) can be written in a manifestly $U(N_c) \times U(N_f - N_c)$ invariant manner as

$$[\chi^\alpha, a^\dagger]_n^m + \theta_l^{\alpha m} W_n^l - \tilde{W}_l^m \tilde{\theta}_n^{\alpha l} = 0 \tag{4.9}$$

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The change from Grassmann coordinates μ and $\tilde{\mu}$ to $\theta, \tilde{\theta}, \nu$ and $\tilde{\nu}$ entails a Jacobian factor, once again computed in [5]

$$\int d\mu \, d\tilde{\mu} = \int d\theta \, d\tilde{\theta} \, d\nu \, d\tilde{\nu} \, |\det_k W|^{-k} |\det_k \tilde{W}|^{-k} \,, \tag{4.10}$$

To integrate over the Grassmannian super-orientation modes ν and $\tilde{\nu}$ we must see how they appear in the matrix model action. In instanton computations, extraneous fermionic zero modes are often lifted by a four-fermi coupling in the action. In the present case such a coupling comes from integrating out the four vector multiplet scalars X. The relevant terms in the vortex matrix model are

$$S_X = -\text{Tr}_k \left(X_a \mathbb{L}[X_a] + X_a \Lambda_a \right) \,, \tag{4.11}$$

where the linear operator $\mathbb{L}$ is given by

$$\mathbb{L}[X_a]^m_n = [a^\dagger, [a, X_a]]^m_n + \frac{1}{2}\{X_a, W + \tilde{W}\}^m_n \tag{4.12}$$

while the adjoint valued fermion bi-linear Λ is (neglecting gauge indices)

$$\Lambda^a = \bar{\chi}_{\dot{\alpha}}(\bar{\sigma}^a)^{\dot{\alpha}\alpha}\chi_\alpha + \chi^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} - \sum_{i=1}^{N_c} \mu_i^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\bar{\mu}^{\dot{\alpha}i} - \sum_{r=1}^{N_f-N_c} \bar{\bar{\mu}}_{\dot{\alpha}r}(\bar{\sigma}^a)^{\dot{\alpha}\alpha}\tilde{\mu}_{\alpha r}$$

$$\equiv \hat{\Lambda}^a - \sum_{i=1}^{N_c} \nu_i^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\bar{\nu}^{\dot{\alpha}i} - \sum_{r=1}^{N_f-N_c} \bar{\nu}_{\dot{\alpha}r}(\bar{\sigma}^a)^{\dot{\alpha}\alpha}\tilde{\nu}_\alpha^r \tag{4.13}$$

where, in the second line, we have invoked the decomposition (4.7) such that the fermi bilinear $\hat{\Lambda}$ does not contain any super-orientation modes ν or $\tilde{\nu}$,

$$\hat{\Lambda}_a = \bar{\chi}_{\dot{\alpha}}(\bar{\sigma}^a)^{\dot{\alpha}\alpha}\chi_\alpha + \chi^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} - \theta^\alpha(\sigma^a)_{\alpha\dot{\alpha}}W\bar{\theta}^{\dot{\alpha}} - \bar{\bar{\theta}}_{\dot{\alpha}}(\bar{\sigma}^a)^{\dot{\alpha}\alpha}\tilde{W}\tilde{\theta}_\alpha \tag{4.14}$$

Integrating out the X fields gives schematically $\mathbb{L}X \sim \Lambda$ which, upon substitution back into the action, yields the appropriate four-fermi terms which may saturate the integration over ν. However, one of the key insights of the analysis of [5] is that it pays dividends to keep the X fields rather than to write them in terms of fermi bi-linears Λ. Indeed, performing the integration over the $k(N_c - k)$ variables ν and $k(N_f - N_c - k)$ variables $\tilde{\nu}$ yields

$$\int d\nu \, d\tilde{\nu} \, \exp\left(-\text{Tr}_k \sum_{i=1}^{N_c} \nu_i^\alpha X_{\alpha\dot{\alpha}}\bar{\nu}^{\dot{\alpha}i} - \text{Tr}_k \sum_{r=1}^{N_f-N_c} \bar{\nu}_r^{\dot{\alpha}}\bar{X}^{\dot{\alpha}\alpha}\tilde{\nu}_\alpha^r\right) = (\det_{2k} X)^{N_f-2k}$$

with $X_{\alpha\dot{\alpha}} = X^a(\sigma_a)_{\alpha\dot{\alpha}}$ and $\bar{X}^{\dot{\alpha}\alpha} = X^a(\bar{\sigma}_a)^{\dot{\alpha}\alpha}$, each a $2k \times 2k$ matrix. Putting everything together, we have an expression for the effective partition function for k instantons,

$$\mathscr{Z}_{k,N_f,N_c} = \frac{C_{k,N_c}C_{k,N_f-N_c}}{\text{Vol}(\text{U}(k))} \int dX \, da \, dW \, d\tilde{W} \, d\chi \, d\theta \, d\tilde{\theta} \, [\Delta] \, (\det_{2k} X_a)^{N_f-2k}$$

$$\times (\det_k W)^{N_c-2k} (\det_k \tilde{W})^{(N_f-N_c)-2k} \exp\left(-\text{Tr}_k(X_a\mathbb{L}[X] + X_a\hat{\Lambda}^a)\right) .$$

where the normalization coefficients $C_{k,N}$, the linear operator $\mathbb{L}$ and the fermi bi-linear $\hat{\Lambda}$ are defined in (4.5), (4.12) and (4.14) respectively. The constraints (3.6) and (4.9) are imposed by the Dirac delta functions, indicated by $[\Delta]$. However, as noted, previously, these constraints are linear in our variables $W, \tilde{W}$ allowing us to perform the integrals. We make the trivial change of variables,

$$W_{\pm} = W \pm \tilde{W} \tag{4.15}$$

so that the bosonic constraint (3.6) reads simply $W_- = [a, a^{\dagger}]$ and is removed by integration over W_-. Indeed, if we are interested in operators of the 2d SCFT which are $U(N_c) \times SU(N_f)$ singlets (as opposed to merely $U(N_c) \times U(N_f - N_c)$ singlets (3.2)) then they may only depend on W_+ in the instanton background. We mimic this procedure for the fermions, defining $\theta_{\pm} = \theta \pm \tilde{\theta}$. The fermionic constraints (4.9) remove one half of our fermionic integrations which we take to be θ_- and $\bar{\theta}_-$, picking up further powers of $\det^k W$ and $\det^k \tilde{W}$. This leaves us with our final, unconstrained result, for the instanton partition function

$$\mathcal{Z}_{k,N_c,N_f} = \frac{C_{k,N_c} C_{k,N_f-N_c}}{2^{kN_f - k^2} \text{Vol}(U(k))} \int dX \, da \, dW_+ d\chi \, d\theta_+ (\det_{2k} X)^{N_f - 2k} \Big(\det_k(W_+ + [a, a^{\dagger}]) \Big)^{N_c - k}$$
$$\times \Big(\det_k(W_+ - [a, a^{\dagger}]) \Big)^{N_f - N_c - k} \exp\Big(-\text{Tr}_k(X_a \mathbb{L}[X] + X_a \hat{\Lambda}^a) \Big)$$

where we remind the reader that all integrations are over the variables and their complex conjugates.

4.1 The partition function for a single instanton

In the case of a single $k = 1$ instanton, the partition function simplifies tremendously, aided by the fact that the adjoint chiral multiplet fields a and χ decouple. The scalar $a = -\sqrt{2\pi r}z$ is simply the position of the instanton, while the complex fermion χ^{α}, which can be rewritten as four real Grassmann parameters ξ^a, are the Goldstino modes of the instanton arising broken $\mathcal{N} = (4, 4)$ supersymmetry.

The hypermultiplet field W_+ is related to the scale size of the instanton

$$W_+ = \rho^2 \tag{4.16}$$

where the exponent follows from noting the scaling dimension $[W_+] = -2$. The fermionic superpartners living in θ_+ are identified with the four superconformal modes of the instanton which we shall denote as η^a. The X determinant is trivially evaluated $\det_2 X = X_a X_a$. Collecting everything together, we have

$$\mathcal{Z}_{1,N_f,N_c} = \frac{2r \, \pi^{N_f} N_c(N_f - N_c)}{N_c! (N_f - N_c)!} \int d^2z \, d^4\xi \, d^4\eta$$
$$\times \int d(\rho^2) \, d^4X \, (X_a X_a)^{N_f - 2} \rho^{2N_f - 4} \exp(-\rho^2 X_a X_a) \tag{4.17}$$

The integral in the final line may be trivially evaluated. We split the X_a, $a = 1, 2, 3, 4$ into radial and polar parts $(u, \hat{\Omega}_3)$. Since we wish to use our effective partition function

to compute correlation functions with $\text{Spin}(4)_R$ charge, we integrate only over the radial direction while leaving the 3-sphere $\hat{\Omega}_3$ intact. (Indeed, in the $N_f \to \infty$ limit a saddle point approximation restricts the integral to $u \sim \rho^{-1}$). Our final result for the partition function for a single instantons is

$$\mathcal{Z}_{1,N_f,N_c} = \frac{2r\,\pi^{N_f}(N_f - 1)!}{(N_c - 1)!(N_f - N_c - 1)!} \int d^2z \frac{d\rho}{\rho^3} d\hat{\Omega}_3 \, d^4\xi d^4\eta \qquad (4.18)$$

which indeed reveals the $AdS_3 \times S^3$ structure as promised. This partition function may be used to compute the instanton contribution to any suitable gauge and flavor singlet correlation function which saturates the 8 supersymmetry and superconformal fermi zero modes.

Acknowledgments

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